MATH2050a Mathematical Analysis I

Exercise 7 suggested Solution

7. Let $f : R \to R$ be continuous at c and let $f(c) > 0$. Show that there exists a neighborhood $V_\delta(c)$ of c such that if $x \in V_\delta(c)$, then $f(x) > 0$.

Solution:

Let $\epsilon = \frac{f(c)}{2} > 0$, Since f is continuous on c, there exists a $\delta > 0$, such that for any $x \in (c-\delta, c+\delta)$, $|f(x) - f(c)| < \frac{f(c)}{2}$. Hence, $\forall x \in (c-\delta, c+\delta)$, $f(x) - f(c) > -\frac{f(c)}{2} > 0$, which implies that $f(x) > 0$.

8. Let $f : R \to R$ be continuous on R and let $S := \{x \in R : f(x) = 0\}$ be the "zero set" of f. If $\{x_n\}$ is in S and $x = lim x_n$, show that $x \in S$.

Solution:

Let $\{x_n\}$ be a sequence in S, and $x = lim(x_n)$. Hence $\forall n \in N$, $f(x_n) = 0$. Since f is continuous on R, $f(x) = lim f(x_n) = 0$, we have $x \in S$.

10. Show that the absolute value function $f(x) := |x|$ is continuous at every point $c \in R$.

Solution:

Fix $c \in R$, for each $\epsilon > 0$, choose $\delta = \epsilon$, for any $x \in (c-\delta, c+\delta)$, $||x| - |c|| < |x - c| < \delta = \epsilon$, hence f is continuous at c. Since c is arbitrary in R, we have f is continuous on R.

12. Suppose that $f : R \to R$ is continuous on R and that $f(r) = 0$ for every rational number r. Prove that $f(x) = 0$ for all $x \in R$.

Solution:

Let $x \in R$, since Q is dense in R, there exists a sequence of rational number $\{q_n\}$ such that $x = lim(x_n)$. By assumption, $f(q_n) = 0, \forall n \in N$. Since f is continuous, $f(x) = lim f(q_n) = 0$. Since x is arbitrary in R, we have $f \equiv 0$.